CANONICAL ISOTOPIES IN EUCLIDEAN SPACE†

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ABSTRACT

Let $G(n,k)$ denote the space (with the compact-open topology) of homeomorphisms of R^n which are fixed on R^k. Theorem 1: $G(n,n-2)$ deforms in $G(n,0)$ to $O(2)$, where $O(2)$ is the orthogonal group. Corollary 2: (for $n=2$) the Kneser theorems. Corollary 3: A Euclidean R^n bundle ξ^n over $S^m(m \leqq \infty)$ which contains an R^{n-2} subbundle ξ^{n-2} is isomorphic, as a $G(n,0)$ bundle, to a Whitney sum $\xi^{n-2} \oplus \xi^2$. Corollary 4: $(n-2)$ stable homeomorphisms of R^n or S^n are $(n-1)$ stable, hence stable if orientation preserving.

Let $G(n,k)$ denote the space (with the compact-open topology) of homeomorphisms of R^n, which are pointwise fixed on R^k. We prove the following theorem.

THEOREM 1. *$G(n,n-2)$ deforms in $G(n,0)$ to $O(2)$, where $O(2)$ is the orthogonal group.*

COROLLARY 2. *The Kneser theorems* [4] *or* [6] *for $n=2$.*

COROLLARY 3. *A Euclidean R^n bundle ξ^n over $S^m(m \leqq \infty)$ which contains an R^{n-2} subbundle ξ^{n-2} is isomorphic, as a $G(n,0)$ bundle, to a Whitney sum $\xi^{n-2} \oplus \xi^2$.*

COROLLARY 4. *$(n-2)$ stable homeomorphisms of R^n or S^n are $(n-1)$ stable, hence stable if orientation preserving.*

Corollary 4 has been given by Cernavskii [1] for $n \neq 4$.

1. Definitions and preliminaries

R^n will denote Euclidean n-space; $R^k \subset R^n$ is identified with $R_k \times \{O\} \subset R^k \times R^{n-k} = R^n$. The ball of radius r in R^n, centered at x, is denoted by $B_r^n(x)$.

† Part of this work represents a portion of the author's Ph.D. thesis at the University of California, Los Angeles, written under the direction of D. S. Gillman and R. C. Kirby.

Received June 6, 1972 and revised form August 15, 1973

If $x = 0$ it is usually omitted. S^1 will denote the unit circle in the plane. $G = G(n)$ denotes the space (with the compact-open topology) of homeomorphisms of R^n. $G(n, k)$ denotes the subspace of those homeomorphisms which are pointwise fixed on R^k. $O(2)$ ($SO(2)$) will denote the (special) orthogonal group, identified with $\mathrm{id}_{|R_{n-2}} \times O(2)$ in $G(n, n-2)$. A basis for the neighborhoods of the identity (id) in G consists of sets of the form $N(C, \varepsilon) = \{h \mid |h(x) - x| < \varepsilon \text{ all } x \in C\}$, where $\varepsilon > 0$, C is compact in R^n, and $|\cdot|$ denotes the usual norm. It is well known that G is a topological transformation group on R^n, and compositions, inverses, and evaluations are continuous. This will be used without further mention in all proofs. We shall occasionally consider functions from R^k to R^m, which we give the compact open topology. For $f, g\colon R^k \to R^m$, we define

$$d(f, g) \text{ on } A = \sup\{|f(x) - g(x)| \mid x \in A\} \text{ for } A \subset R^k.$$

An isotopy h_t of $h \in G$ is a path in G starting at h. We say that an isotopy h_t, for each $h \in A \subset G$ is *canonical* if the function from $A \times I$ into G defined by $(h, t) \to h_t$ is continuous, that is, defines a deformation of A in G. Let J denote the integers; A/B the complement of B in A. We make further definitions as we need them.

2. Some useful lemmas

It will be convenient to denote points in R^n by either rectangular ($(z, r_1, r_2) \in R^{n-2} \times R \times R$) or cylindrical ($(z, r, \theta) \in R^{n-2} \times [0, \infty) \times S^1$) coordinates. For ease of notation we choose a homemorphism $\mu\colon [-\infty, \infty) \to [0, \infty)$, fixed for $r \geqq \frac{1}{2}$. Under this correspondence we may take our cylindrical coordinates in $R^{n-2} \times [-\infty, \infty) \times S^1$ or, or course in $R^{n-2} \times [0, \infty) \times S^1$.

For our first lemma we use the following notation. If $f\colon R^{n-2} \to [-\infty, \infty)$ is continuous, by C_f we mean the set $\{(z, r, \theta) \mid r \leqq f(z)\}$. If f is the constant function $f(z) = r$, we simply write C_r. We note that $C_r = R^{n-2} \times B^2_{\mu(r)}$. If $f_1 < f_2$, $f_3 < f_4$, by $T_t = T_t(f_1; f_2; f_3; f_4)$ we mean the homeomorphism of R^n, fixed on C_{f_1} and off C_{f_4}, which takes the ray $\{z\} \times [-\infty, \infty) \times \{\theta\} = [-\infty, \infty)$ onto itself as follows: T_t is fixed on $[-\infty, f_1(z)]$ and $[f_4(z), \infty)$, taking $[f_1(z), f_2(z)]$ linearly onto $[f_1(z), (1-t)f_2(z) + tf_3(z)]$ and $[f_2(z)\ f_4(z)]$ linearly onto $[(1-t)f_2(z) + tf_3(z), f_4(z)]$. It is readily seen that T_t is continuous in t and f_i, and that it defines an isotopy (of the identity) in $G(n, n-2)$. Let π_1 and π_2 denote projections (in cylindrical coordinates) of R^n on R^{n-2} and $[-\infty, \infty)$ or $[0, \infty)$, depending on how we are taking the radial coordinate.

LEMMA 5. *Let $h \in G(n,n-2)$. Then h is canonically isotopic in $G(n,n-2)$ to h_4, with $d(\pi_1 \circ h_4, \pi_1) \leqq 2$ on R^n and $C_{4j-1} \subset h_4(C_{4j}) \subset C_{4j+1}$ for all $j \in J$. Furthermore, if $h \in O(2)$, then $h_t = h$.*

PROOF. Let $\phi(z) = \min[0; \sup\{r \mid h(B^{n-2}_{\frac{1}{2}}(z) \times B^2_{\mu(r)}) \subset B^{n-2}_1(z) \times B^2_{\mu(0)}\}]$. Set $h_t = T_t(-\infty;\phi;0;1) \circ h \circ T_t^{-1}(-\infty;\phi;0;1)$. Then $d(\pi_1 \circ h_1 \circ \pi_1) \leqq 1$ on C_0. h_1 is now canonically isotoped to h_4 in three steps.

Step 1. For each $i = -1,0,1,2,3,\cdots$ we define $\lambda_i: R^{n-2} \to R$ as follows: Set $\lambda_0 = 0, \lambda_{-1}(z) = \max[1; \sup\{\pi_2 \circ h_1(z',0,\theta) \mid |z'-z| \leqq 1\}]$.

By induction define:

$$\lambda_{2i+1}(z) = \min[-1 + \lambda_{2i}(z); \inf\{\pi_2 \circ h_1(z',\lambda_{2i}(z'),\theta) \mid |z-z'| \leqq 1\}]$$

$$\lambda_{2i+2}(z) = \min[-1 + \lambda_{2i+1}(z); \inf\{\pi_2 \circ h_1^{-1}(z',\lambda_{2i+1}(z'),\theta) \mid |z'-z| \leqq 1\}].$$

Thus $1 + \lambda_{i+1} \leqq \lambda_i$, with $C_{\lambda_{2j+1}} \subset h_1(C_{\lambda_{2j}}) \subset C_{\lambda_{2j-1}}$ for $j \leqq 0$.

Let ψ_t be the isotopy (of the identity) in $G(n, n-2)$ which takes the ray $\{z\} \times [-\infty,\infty) \times \{\theta\} = [-\infty,\infty)$ onto itself. ψ_t is the identity on $[1+\lambda_{-1}(z),\infty)$, it takes $\{-i\}$ to $\{(1-t)(-i) + t\lambda_i(z)\}$ and it is linear on the segments $[-(i+1)-1]$ and $[1, 1+\lambda_{-1}(z)]$. By the choice of λ_i, ψ_t varies continuously with h_1 and the isotopy $h_{1+t} = \psi_t^{-1} \circ h_1 \circ \psi_t$ is canonical. Furthermore $d(\pi_1 h_2, \pi_1) \leqq 1$ on C_0, with $C_{2j-1} \subset h_2(C_{2j}) \subset C_{2j+1}$. If $h_1 \in O(2)$ then $\psi_t = \text{id}$ and $h_{1+t} = h_1$.

Step 2. We canonically isotope h_2 to h_3 by the method described in [2, page 85]. Specifically we define α_1, β, and $\delta_1 \in G(n,n-2)$ by

$$\alpha_1(z,r,\theta) = (z, r-4,\theta) \text{ for } r \leqq -7$$

$$= (z,r,\theta) \quad \text{for } r \geqq -6,$$

and takes $\{z\} \times [-7,-6] \times \{\theta\}$ linearly onto $\{z\} \times [-11,-6] \times \{\theta\}$.

$$\beta(z,r,\theta) = (z,r-4,\theta)$$

$$\delta_1(z,r,\theta) = (z,r+4,\theta) \text{ for } r \leqq -7$$

$$= (z,r,\theta) \qquad \text{for } r \geqq -2, \text{ and takes}$$

$\{z\} \times [-7,-2] \times \{\theta\}$ linearly onto $\{z\} \times [-3,-2] \times \{\theta\}$. Let $\alpha_t(\delta_t)$ be the natural isotopies from the identity to $\alpha_1(\delta_1)$. Set $h'_{2+t} = (\beta \circ h_2) \circ \delta_t \circ (\beta \circ h_2)^{-1} \circ \alpha_t \circ h_2$. This isotopy slides the cylinders $h_2(\dot{C}_{-8})$ near $\dot{C}_{-12}$, along the natural $\{z\} \times [-\infty,\infty) \times \{\theta\}$ fibers, keeping $h_2(\dot{C}_{-4})$ fixed. (Here $(\dot{\ })$ means boundary.) It then slides $\alpha_1 \circ h_2(\dot{C}_{-8})$ back along the fibers provided by $\beta \circ h_2$ until $h'_3(\dot{C}_{-8})$ agrees with $h'_3(\dot{C}_{-4}) = h_2(\dot{C}_{-4})$. We note that $h'_{2+t} = h_2$ off C_{-4}, with

$h'_3 \circ \beta_{|C_{-4}} = \beta \circ h'_{3|C_{-4}}$. Since the isotopy $h'_{2+t} \neq h_2$ for $h_2 \in O(2)$, we modify it by setting $h_{2+t} = (\mathrm{id}'_t)^{-1} h_{2+t}$. Then $h_3 = h'_3$ and (since the $\alpha_t, \delta_t, \beta$ commute with $h \in O(2)$), $h_{2+t} = h_2$ if $h_2 \in O(2)$. Furthermore, since the image of the fibers $(\{z\} \times R^2) \cap C_0$ under h_2 and $\beta \circ h_2$ are within 1 of $\{z\} \times R^2$, we have that $d(\pi_1 \circ h_3, \pi_1) \leqq 2$ on C_{-4}. We also have $C_{4j-1} \subset h_3(C_{4j}) \subset C_{4j+1}$ for $j \leqq -1$.

Step 3. We define $h_4 \in G(n, n-2)$ by $h_{4|C_{-4}} = h_{3|C_{-4}}$ and

$$h_{4|C_{k-4}/C_{k-8}} = \beta^{-k} \circ h_3 \circ \beta^{k}{}_{|C_{k-4}/C_{k-8}}.$$

Then h_4 maps each annulus C_{k-4}/C_{k-8} the same way that h_3 mapped C_{-4}/C_{-8}. h_3 is canonically isotopic to h_4 (since they agree on C_{-4}) and h_4 satisfies the conclusion of the lemma. End of proof.

The following two lemmas are given in [4] for $n = 2$. Since the proofs are identical, they are not given here. However, we emphasize that the space of all bounded homeomorphisms of R^n is *not* contractible, that is, that the M in Lemmas 6 and 7 is necessarily fixed, (refer to [4]).

LEMMA 6. *Let $M > 0$ be given and let $h \in G(n)$ be bounded by M, that is $d(h, id) \leqq M$ on R^n. Then h is canonically isotopic to the identity. Furthermore, if $h = id$, then $h_t = id$.*

LEMMA 7. *Let $M > 0$ be given and let $h \in G(n)$ preserve orientation with $d(\pi \circ h, \pi) \leqq M$ on R^n, where $\pi: R^n \to R^{n-1}$ denotes projection on R^{n-1}. Then h is canonically isotopic to the identity. Furthermore if $h = id$, then $h_t = id$.*

For the next lemma we consider a covering map $\lambda: R^n \to R^n / R^{n-2}$ defined as follows. Let $e: R \to S^1$ be the standard exponential covering map. We define $\lambda': R^n = R^{n-2} \times R \times R \to R^{n-2} \times (0, \infty) \times S^1 = R^n / R^{n-2}$ by $\lambda'(z, r_1, r_2) = (z, \mu(r_1), e(r_2))$; the domain (range) given in rectangular (cylindrical) coordinates. Here $\mu: R \to (0, \infty)$ is the homeomorphism defined in the remarks preceding Lemma 5. Thus λ' wraps the strips $R^{n-2} \times [r_1, r_2] \times R$ around the thickened annuli $R^{n-2} \times (B_{\mu(r_2)} / \mathrm{int}\, B^2_{\mu(r_1)})$. We note that λ' fixes the half line $\{0\} \times [\frac{1}{2}, \infty) \times \{0\}$, and that $\pi = (\pi_1 \circ \lambda', \mu^{-1} \circ \pi_2 \circ \lambda')$, where π_1, π_2 and π are the projections of R^n on R^{n-2}, $(0, \infty)$, and R^{n-1} given in Lemma 7 and the remarks preceding Lemma 5. λ is now taken to be an approximation to λ' which equals λ' off a small compact neighborhood of $x_0 = (0, 1, 0)$, and which is the identity on a smaller compact neighborhood N of x_0.

LEMMA 8. *Let $h \in G(n, n-2)$ fix $x_0 = (0, 1, 0)$. Then $h_{|R^n/R^{n-2}}$ lifts (uniquely) to $\hat{h} \in G(n)$, with $\hat{h}(x_0) = x_0$. Furthermore h is canonically isotopic to $\hat{h}$.*

PROOF. By standard covering space theory $\hat{h}$ uniquely exists. By the definition of λ, we have that $h = \hat{h}$ on a neighborhood of x_0, in fact on the component of $N \cap h^{-1}(N)$ containing x_0. That h is canonically isotopic to $\hat{h}$ now follows from the continuity of $h \to \hat{h}$, the continuity of the function $h \to \operatorname{rad} N \cap h^{-1}(N) = \sup\{r \mid B_r^n(x_0) \subset N \cap h^{-1}(N)\}$, and the following basic observation which is stated without proof as Lemma 9.

LEMMA 9. *If $A \subset G(n)$ is such that for every $h \in A$ we have*

(i) *$h = id$ on a compact neighborhood N_h of x_0 and*

(ii) *$h \to \operatorname{rad}(N_h)$ is continuous,*

then h is canonically isotopic to the identity. Furthermore if $h = id$, then $h_t = id$.

We give the isotopy as follows. Let $\phi_t \in G(n,0)$ for $0 \leqq t < 1$ be defined by $\phi_t(x) = (1-t)x$. Let τ be translation by x_0. Then

$$h_t = \tau \circ \phi^{-1} \circ (\tau^{-1} \circ h \circ \tau) \circ \phi_t \circ \tau^{-1}, \quad \text{for } t < 1$$
$$= \text{id}, \text{ for } t = 1.$$

3. Proof of Theorem 1

Let $h \in G(n, n-2)$ be given. W.l.o.g. assume that h preserves orientation. It suffices to give a canonical isotopy, in $G(n,0)$, of h into $SO(2)$. By Lemma 5, h is canonically isotopic to h_1, with h_1 as h_4 in the conclusion of Lemma 5. Consider $h_1(x_0) = (z, s, \theta_0)$. Let $\rho \in SO(2)$ be rotation by θ_0 about R^{n-2}, that is, $\rho(z,r,\theta) = (z,r,\theta+\theta_0)$. Set $g_1 = \rho^{-1} \circ h_1$. By canonically translating in a small coordinate neighborhood of x_0, we obtain g_2 fixing x_0. We now apply Lemma 8 to g_2, obtaining $\hat{g}_2$, with g_2 canonically isotopic to $\hat{g}_2$. But $\hat{g}_2$ satisfies the hypothesis of Lemma 7 (since g_2 was sufficiently small in the radial and R^{n-2} coordinates, with $\pi = (\pi_1 \circ \lambda', \mu^{-1} \circ \pi_2 \circ \lambda')$ and it is canonically isotopic to the identity. Composing with ρ gives a canonical isotopy from g_1 to ρ. Since the isotopy h_t from h to ρ may move the origin, we translate by $-h_t(0)$ to obtain the isotopy in $G(n,0)$. If $h \in SO(2)$, then $h = h_1 = \rho$, and $g_1 = g_2 = \hat{g} = \text{id}$.
End of proof.

4. Remarks on the corollaries

REMARK 1. Theorem 1 remains valid with S^n, S^{n-2} in place of R^n, R^{n-2}.

REMARK 2. For $n = 2$, we derive the Kneser theorems, namely that $O(2)$

(respectively, $O(3)$ is a strong deformation retract of the homeomorphism group of R^2 (respectively, S^2). See [4].

REMARK 3. Let $H(n, n-2)$ denote those homeomorphisms which are invariant on R^{n-2}, and fix 0. Then $H(n, n-2) = G(n, n-2) \times G(n-2, 0)$. This gives Theorem 1′.

THEOREM 1′. *$H(n, n-2)$ deforms in $G(n, 0)$ to $G(n-2, 0) \times O(2)$ which then gives Corollary* 3.

By an isomorphism (in Corollary 3) we mean a homeomorphism between the total spaces which preserves fibers and is the identity on the zero section, that is, a $G(n, 0)$ bundle equivalence as in [7].

REMARK 4. Corollary 4 is just the non-canonical version of Theorem 1 with Remark 1.

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